

ON THE RANKS OF THE THIRD SECANT VARIETY OF SEGRE-VERONESE EMBEDDINGS

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ABSTRACT. We give an upper bound for the rank of the border rank 3 partially symmetric tensors. In the special case of border rank 3 tensors of k factors without any symmetry (Segre case) we can show that all ranks among 3 and $k - 1$ arise.

INTRODUCTION

In this paper we deal with the problem of finding a bound for the minimum integer r needed to write a given tensor T as a linear combination of r indecomposable tensors. Such a minimum number is now known under the name of *rank of T* (Definition 1.4). In order to be as general as possible we will consider the tensor T to be partially symmetric, i.e.

$$T \in S^{d_1} V_1 \otimes \cdots \otimes S^{d_k} V_k \quad (1)$$

where the d_i 's are positive integers and V_i 's are finite dimensional vector spaces defined over an algebraically closed field K . The decomposition that will give us the rank of such a tensor will be of the following type:

$$T = \sum_{i=1}^r \lambda_i v_{1,i}^{\otimes d_1} \otimes \cdots \otimes v_{k,i}^{\otimes d_k} \quad (2)$$

where $\lambda_i \in K$ and $v_{j,i} \in V_j$, $i = 1, \dots, r$ and $j = 1, \dots, k$.

Another very interesting and useful notion of “rank” is the minimum r such that a tensor T can be written as a limit of a sequence of rank r tensors. This last integer is called the *border rank of T* (Definition 1.5) and clearly it can be strictly smaller than the rank of T (Remark 1.6). It has become a common technique to fix a class of tensors of given border rank and then study all the possible ranks arising in that family (cf. [6, 3, 9, 13]). The rank of tensors of border rank 2 is well known (cf. [6] for symmetric tensors, [2] for tensors without any symmetry, [4] for partially symmetric tensors). The first not completely classified case is the one of border rank 3 tensors. In [6, Theorem 37] the rank of any symmetric order d tensor of border rank 3 has been computed and it is shown that the maximum rank reached is $2d - 1$. In the present paper, Theorem 1.7, we prove that the rank of partially symmetric tensors T as in (1) of border rank 3 can be at most

$$r(T) \leq -1 + \sum_{i=1}^k 2d_i.$$

In [9, Theorem 1.8] J. Buczyński and J.M. Landsberg described the cases in which the inequality in Theorem 1.7 is an equality: when $k = 3$ and $d_1 = d_2 = d_3 = 1$ they show that there is an element of rank 5. All ranks for border rank 3 partially symmetric tensors are described in [8] when $k = 3$, $d_1 = d_2 = d_3 = 1$ and $n_i = 1$ for at least one integer i . Therefore our Theorem 1.7 is the natural extension of the two extreme cases (tensors without any symmetry where $d_i = 1$ for all $i = 1, \dots, k$ and totally symmetric case where $k = 1$).

In the special case of tensors without any symmetry, i.e. $T \in V_1 \otimes \cdots \otimes V_k$, we will be able to show, in Theorem 1.8, that any rank between 3 and $k - 1$ arises among border rank 3 tensors. In the proof of this theorem we will describe the structure of our solutions: they are all obtained from $(\mathbb{P}^1)^k$ taking as a border scheme a degree 3 connected curvilinear scheme (Proposition 3.1 gives the case of rank $k - 1$ when $k \geq 4$ and the other cases follow taking a smaller number of factors).

In [5] we defined the notion of curvilinear rank for symmetric tensors to be the minimum length of a curvilinear scheme whose span contains a given symmetric tensor. We can extend some of the ideas in [5] and some of those used in our proof of Theorem 1.7 to the case of partially symmetric

tensors and prove that, if a partially symmetric tensor is contained in the span of a special degree c curvilinear scheme with α components, the rank of this tensor is bounded by $2\alpha + c \left(-1 + \sum_{i=1}^k d_i \right)$ (cf. Theorem 1.10).

1. NOTATION, DEFINITIONS AND STATEMENTS

In this section we introduce the basic geometric tools that we will use all along the paper.

Notation 1.1. We indicate with

$$\nu : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \rightarrow \mathbb{P}^M,$$

$M = \left(\prod_{i=1}^k (n_i + 1) \right) - 1$ the *Segre embedding* of the multi-projective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, i.e. the embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ by the complete linear system $|\mathcal{O}_{\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}}(1, \dots, 1)|$.

For each $i \in \{1, \dots, k\}$ let

$$\pi_i : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \rightarrow \mathbb{P}^{n_i}$$

denote the projection onto the i -th factor.

Let

$$\tau_i : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \rightarrow \mathbb{P}^{n_1} \times \cdots \times \hat{\mathbb{P}}^{n_i} \times \cdots \times \mathbb{P}^{n_k}$$

denote the projection onto all the factors different from \mathbb{P}^{n_i} .

Let $\varepsilon_i \in \mathbb{N}^k$ be the k -ple of integers $\varepsilon_i = (0, \dots, 1, \dots, 0)$ with 1 only in the i -th position. We say that a curve $C \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ has *multi-degree* (a_1, \dots, a_k) if for all $i = 1, \dots, k$ the line bundle $\mathcal{O}_C(\varepsilon_i)$ has degree a_i .

We say that a morphism $h : \mathbb{P}^1 \rightarrow \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ has *multi-degree* (a_1, \dots, a_k) if, for all $i = 1, \dots, k$:

$$h^*(\mathcal{O}_{\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}}(\varepsilon_i)) \cong \mathcal{O}_{\mathbb{P}^1}(a_i).$$

Let

$$\nu_{d_1, \dots, d_k} : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \rightarrow \mathbb{P}^N,$$

$N = \left(\prod_{i=1}^k \binom{d_i + n_i}{n_i} \right) - 1$ denote the Segre-Veronese embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ of multi-degree (d_1, \dots, d_k) and define

$$X := \nu_{d_1, \dots, d_k}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$$

to be the *Segre-Veronese variety*.

The name “Segre-Veronese” is classically due to the fact that when the d_i ’s are all equal to 1, then the variety X is called “Segre variety”; while when $k = 1$ then X is known to be a “Veronese variety”.

Remark 1.2. An element of X is the projective class of an indecomposable partially symmetric tensor $T \in S^{d_1}V_1 \otimes \cdots \otimes S^{d_k}V_k$ where $\mathbb{P}(V_i) = \mathbb{P}^{n_i}$. More precisely $p \in X$ if there exists $T \in S^{d_1}V_1 \otimes \cdots \otimes S^{d_k}V_k$ such that $p = [T] = [v_{1,i}^{\otimes d_1} \otimes \cdots \otimes v_{k,i}^{\otimes d_k}]$ with $[v_{j,i}] \in \mathbb{P}^{n_i}$.

Definition 1.3. The s -th secant variety $\sigma_s(X)$ of X is the Zariski closure of the union of all s -secant \mathbb{P}^{s-1} to X . The tangential variety $\tau(X)$ is the Zariski closure of the union of all tangent lines to X .

Observe that

$$X = \sigma_1(X) \subset \tau(X) \subset \sigma_2(X) \subset \cdots \subset \sigma_{s-1}(X) \subset \sigma_s(X) \subset \cdots \subset \mathbb{P}^N. \quad (3)$$

Definition 1.4. The X -rank $r_X(p)$ of an element $p \in \mathbb{P}^N$ is the minimum integer s such that there exist a $\mathbb{P}^{s-1} \subset \mathbb{P}^N$ which is s -secant to X and containing p .

We indicate with $\mathcal{S}(p)$ the set of sets of points of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ “evincing” the X -rank of $p \in \mathbb{P}^N$, i.e.

$$\mathcal{S}(p) := \{ \{x_1, \dots, x_s\} \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \mid r_X(p) = s \text{ and } p \in \langle \nu_{d_1, \dots, d_k}(x_1), \dots, \nu_{d_1, \dots, d_k}(x_s) \rangle \}.$$

Definition 1.5. The X -border rank $br_X(p)$ of an element $p \in \mathbb{P}^N$ is the minimum integer s such that $p \in \sigma_s(X)$.

Remark 1.6. For any $p \in \mathbb{P}^N = \mathbb{P}(S^{d_1}V_1 \otimes \cdots \otimes S^{d_k}V_k)$ we obviously have that $br_X(p) \leq r_X(P)$. In fact $p \in \mathbb{P}^N$ of rank r is such that there exist a tensor $T \in S^{d_1}V_1 \otimes \cdots \otimes S^{d_k}V_k$ that can be minimally written as in (2); while an element $p \in \mathbb{P}^N$ has border rank s if and only there exist a sequence of rank r tensors $T_i \in S^{d_1}V_1 \otimes \cdots \otimes S^{d_k}V_k$ such that $\lim_{i \rightarrow \infty} T_i = T$ and $p = [T]$.

The first result that we prove in Section 2 is an upper bound for the rank of points in $\sigma_3(X)$.

Theorem 1.7. *The rank of an element $p \in \sigma_3(X)$ is $r_X(p) \leq -1 + \sum_{i=1}^k 2d_i$.*

In the case $d_i = 1$ for all $i = 1, \dots, k$, i.e. if X is the Segre variety, we fill in all low ranks with points of $\sigma_3(X) \setminus \sigma_2(X)$. In Section 3 we prove the following result.

Theorem 1.8. *Assume $k \geq 3$ and let $X \subset \mathbb{P}^M$ be the Segre variety of k factors. For each $x \in \{3, \dots, k-1\}$ there is $p \in \sigma_3(X) \setminus \sigma_2(X)$ with $r_X(p) = x$.*

As remarked in the Introduction, we can extend some of the ideas of [5] on the notion of curvilinear rank to some ideas used in our proof of Theorem 1.7 to the case of partially symmetric tensors.

Definition 1.9. A scheme $Z \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is *curvilinear* if it is a finite union of disjoint schemes of the form $\mathcal{O}_{C_i, P_i}/m_{P_i}^{e_i}$ for smooth points $p_i \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ on reduced curves $C_i \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, or equivalently that the tangent space at each connected component of Z supported at the P_i 's has Zariski dimension ≤ 1 . We define the *curvilinear rank* $\text{Cr}(p)$ of a point $p \in \mathbb{P}^N$ as:

$$\text{Cr}(p) := \min \{ \deg(Z) \mid \nu_{d_1, \dots, d_k}(Z) \subset X, Z \text{ curvilinear}, p \in \langle \nu_{d_1, \dots, d_k}(Z) \rangle \}.$$

In Section 4 we prove the following result.

Theorem 1.10. *If there exists a degree c curvilinear scheme $Z \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ such that $p \in \langle \nu_{d_1, \dots, d_k}(Z) \rangle$ and Z has α connected components, each of them mapped by ν_{d_1, \dots, d_k} into a linearly independent zero-dimensional sub-scheme of \mathbb{P}^N , then $r_X(p) \leq 2\alpha + c \left(-1 + \sum_{i=1}^k d_i \right)$.*

2. PROOF OF THEOREM 1.7

Remark 2.1. Fix a degree 3 connected curvilinear scheme $E \subset \mathbb{P}^2$ not contained in a line and a point $u \in \mathbb{P}^1$. The scheme E is contained in a smooth conic. Hence there is an embedding $f: \mathbb{P}^1 \rightarrow \mathbb{P}^2$ with $f(\mathbb{P}^1) = C$ and $f(3u) = E$.

Remark 2.2. For any couple of points $u, o_i \in \mathbb{P}^1$, there is an isomorphism $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ such that $f(u) = o_i$. For any such f we have $f(3u) = 3o_i$.

Remark 2.3. Fix two points $u, o_i \in \mathbb{P}^1$. There is a morphism $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ such that $f(u) = o_i$, f is ramified at u and $\deg(f) = 2$, i.e. $f^*(\mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathcal{O}_{\mathbb{P}^1}(2)$. Since $\deg(f) = 2$, f has only order 1 ramification at u . Thus $f(3u) = 2o_i$ (as schemes).

We recall the following lemma proved in [4, Lemma 3.3].

Lemma 2.4 (Autarky). *Let $p \in \langle X \rangle$ with X being the Segre-Veronese variety of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. If there exist \mathbb{P}^{m_i} , $i = 1, \dots, k$, with $m_i \leq n_i$, such that $p \in \langle \nu_{d_1, \dots, d_k}(\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_k}) \rangle$, then the X -rank of p is the same as the Y -rank of p where Y is the Segre-Veronese embedding $\nu_{d_1, \dots, d_k}(\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_k})$.*

Corollary 2.5. *Let $\Gamma \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be the minimal 0-dimensional scheme such that $p \in \langle \nu_{d_1, \dots, d_k}(\Gamma) \rangle$, then the X -rank of p is equal to its Y -rank where Y is the Segre-Veronese embedding of $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_k}$ where each $m_i = \dim \langle \pi_i(\Gamma) \rangle \leq \deg(\pi_i(\Gamma)) - 1$ (π_i as in Notation 1.1); moreover if there exists an index i such that $\deg(\pi_i(\Gamma)) = 1$, then we can take Y to be the Segre-Veronese embedding of $\mathbb{P}^{m_1} \times \cdots \times \hat{\mathbb{P}}^{m_i} \times \cdots \times \mathbb{P}^{m_k}$.*

Proof. Consider the projections $\pi_i: \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ onto the i -th factor \mathbb{P}^{n_i} as in Notation 1.1. It may happen that $\deg(\pi_i(\Gamma))$ can be any value from 1 to $\deg(\Gamma)$.

By the just recalled Autarky Lemma, we may assume that each $\pi_i(\Gamma)$ spans the whole \mathbb{P}^{n_i} . Therefore if there is an index $i \in \{1, \dots, k\}$ such that $\deg(\pi_i(\Gamma)) = 1$ we can take $p \in \mathbb{P}^{n_1} \times \cdots \times \hat{\mathbb{P}}^{n_i} \times \cdots \times \mathbb{P}^{n_k}$. Moreover the autarchic fact that we can assume \mathbb{P}^{n_i} to be $\langle \pi_i(\Gamma) \rangle$ implies that we can replace each \mathbb{P}^{n_i} with $\mathbb{P}^{\dim \langle \pi_i(\Gamma) \rangle}$ and clearly $\dim \langle \pi_i(\Gamma) \rangle \leq \deg(\pi_i(\Gamma))$. \square

Proof of Theorem 1.7: Because of the filtration of secants varieties (3), for a given element $p \in \sigma_3(X)$, it may happen that either $p \in X$, or $p \in \sigma_2(X) \setminus X$ or $p \in \sigma_3(X) \setminus \sigma_2(X)$. We distinguish among these cases.

- (1) If $p \in X$, then $r_X(p) = 1$.
- (2) If $p \in \sigma_2(X) \setminus X$ then either p lies on a honest be-secant line to X (and in this case obviously $r_X(p) = 2$) or p belongs to certain tangent line to X . In this latter case, consider the minimum number h of factors containing such a tangent line, i.e. $h \leq k$ is the minimum integer such that $p \in \langle \nu_{d_1, \dots, d_h}(\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_h}) \rangle$ (maybe reordering factors). In [4, Theorem 3.1] we proved that, if this is the case, then $r_X(p) = \sum_{i=1}^h d_i$.
- (3) From now on we assume that $p \in \sigma_3(X) \setminus \sigma_2(X)$. By [9, Theorem 1.2] there are a short list of zero-dimensional schemes $\Gamma \subset \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ such that $p \in \langle \nu_{d_1, \dots, d_k}(\Gamma) \rangle$, therefore, in order to prove Theorem 1.7, it is sufficient to bound the rank of the points in $\langle \nu_{d_1, \dots, d_k}(\Gamma) \rangle$ for each Γ in their list.

Since $p \in \sigma_3(X) \setminus \sigma_2(X)$, The possibilities for Γ are only the following: either Γ is a smooth degree 3 zero-dimensional scheme (case (3a) below), or it is the union of a degree 2 scheme supported at one point and a simple point (case (3b)), or it is a curvilinear degree 3 scheme (case (3c)) or, finally, a very particular degree 4 scheme with 2 connected components of degree 2 (case (3d)).

- (3a) If Γ is a set of 3 distinct points, then obviously $r_X(p) = 3$ ([9, Case (i), Theorem 1.2]).
- (3b) If Γ is a disjoint union of a simple point a and a degree 2 connected scheme ([9, Case (ii), Theorem 1.2]), then there is a point q on a tangent line to X such that $p \in \langle \{\nu_{d_1, \dots, d_k}(a), q\} \rangle$. Hence $r_X(p) \leq 1 + r_X(q) \leq 1 + \sum_{i=1}^k d_i$ (for the rank on the tangential variety of X see [2]). Since $d_i > 0$ for all i 's and $k \geq 2$, then $1 + \sum_{i=1}^k d_i \leq -1 + \sum_{i=1}^k 2d_i$.
- (3c) Assume $\deg(\Gamma) = 3$ and that Γ is connected ([9, Case (iii), Theorem 1.2]) supported at a point $\{o\} := \Gamma_{\text{red}}$. Since the case $k = 1$ is true by [6, Theorem 37], we can prove the theorem by using induction on k , with the case $k = 1$ as the starting case. Since $\deg(\Gamma) = 3$, by Corollary 2.5, we can assume that p belongs to a Segre-Veronese variety of k factors all of them being either \mathbb{P}^1 's or \mathbb{P}^2 's, i.e., after having reordered the factors,

$$p \in \nu_{d_1, \dots, d_k}(\mathbb{P}^1 \times \dots \times \mathbb{P}^1 \times \mathbb{P}^2 \times \dots \times \mathbb{P}^2).$$

The \mathbb{P}^1 's correspond to the cases in which either $\deg(\pi_i(\Gamma)) = 3$ and $\dim \langle \pi_i(\Gamma) \rangle = 1$ (i.e. $\pi_i(\Gamma)$ is contained in a line of the original \mathbb{P}^{n_i}), or $\deg(\pi_i(\Gamma)) = 2$ (notice that in this case $\pi_i|_{\Gamma}$ is not an embedding). The \mathbb{P}^2 's correspond to the cases in which $\dim \langle \pi_i(\Gamma) \rangle = 2$, $= \deg(\pi_i(\Gamma)) = 3$. Finally we can exclude all the cases in which $\deg(\pi_i(\Gamma)) = 1$ because, again by Corollary 2.5, we would have that p belongs to a Segre-Veronese variety of less factors and then this won't give the highest bound for the rank of p .

Now fix a point $u \in \mathbb{P}^1$. By Remarks 2.1, 2.2 and 2.3 there is

$$f_i : \mathbb{P}^1 \rightarrow \mathbb{P}^{n_i} \text{ with } f_i(3u) = \pi_i(\Gamma). \quad (4)$$

Consider the map

$$f = (f_1, \dots, f_k) : \mathbb{P}^1 \rightarrow \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}.$$

We have $f(u) = \{o\}$ and $\pi_i(f(3u)) = f_i(3u) = \pi_i(\Gamma)$. Since $\pi_i(f(3u)) = \pi_i(\Gamma)$ for all i 's, the universal property of products gives $f(3u) = \Gamma$. The map f has multi-degree (a_1, \dots, a_k) where $a_i := 1$ if $n_i = 1$ and $\deg(\pi_i(\Gamma)) = 3$, and $a_i = 2$ in all other cases. Notice that f_i is an embedding if $\deg(\pi_i(\Gamma)) \neq 2$. Since $\deg(\pi_i(\Gamma)) = 2$ if and only if $\pi_i^{-1}(o_i)$ contains the line spanned by the degree 2 sub-scheme of Γ , we have $\deg(\pi_i(\Gamma)) = 2$ for at most one index i . Since $k \geq 2$, f is an embedding. Set

$$C := \nu_{d_1, \dots, d_k}(f(\mathbb{P}^1)) \text{ and } Z := \nu_{d_1, \dots, d_k}(\Gamma).$$

The curve C is smooth and rational of degree $\delta := \sum_{i=1}^k a_i d_i$. Note that $\delta \leq \sum_{i=1}^k 2d_i$. Hence to prove Theorem 1.7 in this case it is sufficient to prove that $r_C(p) \leq \delta - 1$ because clearly $r_C(p) \leq r_X(p)$ since $C \subset X$.

By assumption $p \in \langle Z \rangle$. Since $p \notin \sigma_2(X)$, $\langle Z \rangle$ is not a line of \mathbb{P}^N . Hence $\langle Z \rangle$ is a plane. Since C is a degree δ smooth rational curve, we have $\dim \langle C \rangle \leq \delta$. By [13, Proposition 5.1] we have $r_C(p) \leq \dim \langle C \rangle$. Hence it is sufficient to prove the case $\delta = \dim \langle C \rangle$, i.e. we may assume that C is a rational normal curve in its linear span. If $\delta \geq 4$, since Z is connected and of degree 3, by Sylvester's theorem (cf. [10]) we have p has C -border rank 3 and $r_C(p) = \delta - 1$, concluding the proof in this case.

If $\delta \leq 3$, we have $\sigma_2(C) = \langle C \rangle$ and hence $p \in \sigma_2(X)$, contradicting $p \in \sigma_3(X) \setminus \sigma_2(X)$.

- (3d) Assume that Γ has degree 4 ([9, Case (iv), Theorem 1.2]). J. Buczyński and J.M. Landsberg show that p belongs to the span of two tangent lines to X whose set theoretic intersections with X span a line which is contained in X . This means that $\Gamma = v \sqcup w$ with v, w degree 2 reduced zero-dimensional schemes with support contained in a line $L \subset \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ and moreover that the multi-degree of L is ε_i for some $i = 1, \dots, k$ (cfr. Notation 1.1). This case occurs only when $d_i = 1$, i.e. when $\nu_{d_1, \dots, d_k}(L) = \tilde{L}$ is a line.

Observe that $\tilde{v} := \nu_{d_1, \dots, d_k}(v)$ and $\tilde{w} := \nu_{d_1, \dots, d_k}(w)$ are two tangent vectors to X . In [2, Theorem 1] we prove that the X -rank of a point $p \in T_o(X)$ for certain point $o = (o_1, \dots, o_k) \in X$, is the minimum number $\eta_X(p)$ for which there exist $E \subseteq \{1, \dots, k\}$ such that $\sharp(E) = \eta_X(p)$ and $T_o(X) \subseteq \langle \cup_{i \in E} Y_{o,i} \rangle$ where $Y_{o,i}$ the n_i -dimensional linear subspace obtained by fixing all coordinates $j \in \{1, \dots, k\} \setminus \{i\}$ equal to $o_j \in \mathbb{P}^{n_i}$. Let I and J be the sets playing the role of E for \tilde{v} and \tilde{w} respectively and set $I' = I \setminus \{i\}$ (meaning that $I' = I$ if $i \notin I$ and $I' = I \setminus \{i\}$ otherwise) and $J' = J \setminus \{i\}$. Now take

$$\alpha := \sum_{j \in I'} d_j + \sum_{j \in J'} d_j + d_i$$

and note that $\alpha \leq -1 + \sum_{h=1}^k 2d_h$, therefore if we prove that $r_X(p) \leq \alpha$ we are done. Let $D_j \subset \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$, $j \in I'$, be the line of multi-degree ε_j containing $\pi_j(v)$, and let T_j , $j \in J'$, be the line of X of multi-degree ε_j containing $\pi_j(w)$. The curve $L \cup \left(\bigcup_{j \in I'} D_j \right)$ contains v and the curve $L \cup \left(\bigcup_{j \in J'} T_j \right)$ contains w . Hence the curve

$$T := L \cup \left(\bigcup_{j \in I'} D_j \right) \cup \left(\bigcup_{j \in J'} T_j \right)$$

is a reduced and connected curve containing Γ . Since $p \in \langle \nu_{d_1, \dots, d_k}(\Gamma) \rangle$, we have that if we call $\tilde{T} := \nu_{d_1, \dots, d_k}(T)$ then $p \in \langle \tilde{T} \rangle$ and $r_X(p) \leq r_{\tilde{T}}(p)$. The curve \tilde{T} is a smooth and connected curve whose irreducible components are smooth rational curves and with $\deg(\tilde{T}) = \alpha$. Hence $\dim \langle \tilde{T} \rangle \leq \alpha$. Since \tilde{T} is reduced and connected, as in [13, Proposition 4.1] we get $r_{\tilde{T}}(p) \leq \alpha$. Summing up $r_X(p) \leq r_{\tilde{T}}(p) \leq \alpha \leq -1 + \sum_{h=1}^k 2d_h$. \square

3. PROOF OF THEOREM 1.8

Autarky Lemma (proved in [4, Lemma 3.3] and recalled here in Lemma 2.4) is true also for the border rank ([8, Proposition 2.1]). This allows to formulate the analog of Corollary 2.5 for border rank. Therefore, in order to prove Theorem 1.8 we can limit ourselves to the study of the case $n_i = 1$ for all i 's. This is the reason why in this section we will always work with the Segre variety of \mathbb{P}^1 's. Let

$$\nu : (\mathbb{P}^1)^k \rightarrow \mathbb{P}^r, \quad r = 2^k - 1$$

be the Segre embedding of k copies of \mathbb{P}^1 's and $X := \nu((\mathbb{P}^1)^k)$; and let

$$\nu' : (\mathbb{P}^1)^{k-1} \rightarrow \mathbb{P}^{r'}, \quad r' = 2^{k-1} - 1$$

be the the Segre variety of $k-1$ copies of \mathbb{P}^1 's and $X' := \nu'((\mathbb{P}^1)^{k-1})$.

Proposition 3.1. *Assume $k \geq 3$. Let $\Gamma \subset (\mathbb{P}^1)^k$ be a degree 3 connected curvilinear scheme such that $\deg(\pi_i(\Gamma)) = 3$ for all i 's, and let β be the only degree 2 sub-scheme of Γ . For all $p \in \langle \nu(\Gamma) \rangle \setminus \langle \nu(\beta) \rangle$ we have that*

- (a) *if $k = 3$, then $2 \leq r_X(p) \leq 3$ and $r_X(p) = 2$ if p is general in $\langle \nu(\Gamma) \rangle$;*

(b) if $k \geq 4$, then $r_X(p) = k - 1$.

Proof. Since $\Gamma \subset (\mathbb{P}^1)^k$ is connected, it has support on only one point; all along this proof we set

$$o := \text{Supp}(\Gamma) \in (\mathbb{P}^1)^k. \quad (5)$$

First of all recall that in step (3a) of the proof of Theorem 1.7 we obtained an embedding $f = (f_1, \dots, f_k)$ with $f_i : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ an isomorphism (see (4)); moreover we can fix a point $u \in \mathbb{P}^1$ such that $f(u) = o$ and $\Gamma = f(3u)$. We proved that

$$C := \nu(f(\mathbb{P}^1))$$

is a degree k rational normal curve in its linear span. Obviously

$$r_X(p) \leq r_C(p).$$

If $k \geq 4$ Sylvester's theorem implies $r_C(p) = k - 1$. Now assume $k = 3$. Since a degree 3 rational plane curve has a unique singular point, for any $q \in \langle C \rangle$ there is a unique line $L \subset \langle C \rangle = \mathbb{P}^3$ with $\deg(L \cap C) = 2$. Thus $r_C(p) = 2$ (resp. $r_C(p) = 3$) if and only if $p \notin \tau(C)$ (resp. $p \in \tau(C)$, cfr. Definition 1.3). Since $\tau(C)$ is a degree 4 surface, by Riemann-Hurwitz, we see that both cases occur and that $r_C(p) = 2$ (and hence $r_X(p) \leq 2$ if p is general in $\langle \nu(\Gamma) \rangle$).

Claim 1. *Let the point $o \in (\mathbb{P}^1)^k$ be, as in (5), the support of Γ . Fix any $F \in |\mathcal{O}_{(\mathbb{P}^1)^k}(\varepsilon_k)|$ such that $o \notin F$. Then $\langle \nu(\Gamma) \rangle \cap \langle \nu(F) \rangle = \emptyset$.*

Proof of Claim 1. It is sufficient to show that $h^0(\mathcal{I}_{F \cup \Gamma}(1, \dots, 1)) = h^0(\mathcal{I}_F(1, \dots, 1)) - 3$, i.e. $h^0(\mathcal{I}_\Gamma(1, \dots, 1, 0)) = h^0(\mathcal{O}_{(\mathbb{P}^1)^k}(1, \dots, 1, 0)) - 3$. This is true because f_1 and f_2 (recalled at the beginning of the proof this Proposition 3.1 and introduced in (4)) are isomorphisms. \square

- (a) Assume $k = 3$. Since $r_X(p) \leq r_C(p) \leq 3$ and $r_C(p) = 2$ for a general p in $\langle \nu(\Gamma) \rangle$, we only need to prove that $r_X(p) > 1$. The case $r_X(p) = 1$ corresponds to a completely decomposable tensor: $p = \nu(q)$ for some $q \in (\mathbb{P}^1)^3$. Clearly $r_X(\nu(o)) = 1$ but $o \in \langle \beta \rangle$ then, since we took $p \in \langle \nu(\Gamma) \rangle \setminus \langle \nu(\beta) \rangle$, we have $p \neq \nu(o)$ and in particular $q \neq o$. In this case we can add q to Γ and get that $h^1(\mathcal{I}_{q \cup \Gamma}(1, 1, 1)) > 0$ by [1, Lemma 1]. Since $\deg(f_i(\Gamma)) = 3$, for all i 's, every point of $\langle \beta \rangle \setminus \{o\}$ has rank 2. Since $q := (q_1, q_2, q_3) \neq o := (o_1, o_2, o_3)$ we have $q_i \neq o_i$ for some i , say for $i = 3$. Take $F \in |\mathcal{O}_{(\mathbb{P}^1)^3}(\varepsilon_3)|$ such that $q \in F$ and $o \notin F$. Hence $F \cap (\Gamma \cup \{q\}) = \{q\}$. We have $h^1(F, \mathcal{I}_{q, F}(1, 1, 1)) = 0$, because $\mathcal{O}_{(\mathbb{P}^1)^3}(1, 1, 1)$ is spanned. Claim 1 gives $h^1(\mathcal{I}_\Gamma(1, 1, 0)) = 0$. The residual exact sequence of F in $(\mathbb{P}^1)^3$ gives $h^1(\mathcal{I}_{\Gamma \cup q}(1, 1, 1)) = 0$, a contradiction.
- (b) From now on we assume $k \geq 4$ and that Proposition 3.1 is true for a smaller number of factors. Since $X \supset C$, we have $r_X(p) \leq k - 1$ (in fact, as we already recalled above, $r_C(p) = k - 1$ by Sylvester's theorem). We need to prove that we actually have an equality, so we assume $r_X(p) \leq k - 2$ and we will get a contradiction.

Take a set of points $S \in \mathcal{S}(p)$ of $(\mathbb{P}^1)^k$ evincing the X -rank of p (see Definition 1.4) and consider $v = (v_1, \dots, v_k) \in S \subset (\mathbb{P}^1)^k$ to be a point appearing in a decomposition of p . We can always assume that, if $o = (o_1, \dots, o_k)$, then $v_k \neq o_k$: such a $v \in S \subset \mathcal{S}(p)$ exists because, by Autarky (here recalled in Lemma 2.4), no element of $\mathcal{S}(p)$ is contained in $(\mathbb{P}^1)^{k-1} \times \{o_k\}$.

Consider the pre-image

$$D := \pi_k^{-1}(v_k).$$

Clearly by construction $o \notin D$ hence for any $q \in (\mathbb{P}^1)^k \setminus D$ we have $h^1(\mathcal{I}_{q \cup D}(1, \dots, 1)) = h^1(\mathcal{I}_q(1, \dots, 1, 0)) = 0$, because $\mathcal{O}_{(\mathbb{P}^1)^k}(1, \dots, 1, 0)$ is globally generated. This implies that $\langle \nu(D) \rangle$ intersects X only in $\nu(D)$.

Now consider

$$\ell : \mathbb{P}^r \setminus \langle \nu(D) \rangle$$

the linear projection from $\langle \nu(D) \rangle$. Since $p \notin \langle \nu(D) \rangle$ (Claim 1), ℓ is defined at p . Moreover the map ℓ induces a rational map $\nu((\mathbb{P}^1)^k \setminus D) \rightarrow \nu((\mathbb{P}^1)^k)$, which extends to the projection $\tau_k : (\mathbb{P}^1)^k \rightarrow (\mathbb{P}^1)^{k-1}$ defined in Notation 1.1. We have

$$\ell \circ \nu = \nu'.$$

Since $o \notin D$, $\ell(\langle \Gamma \rangle) = \langle \nu'(\Gamma') \rangle$, where $\Gamma' = \tau_k(\Gamma)$. Hence $p' := \ell(p) \in \langle \nu'(\Gamma') \rangle$. By [2] every element of $\langle \nu'(\beta) \rangle \setminus \nu'(o')$, with $o' := \tau_k(o)$, has X' -rank $k - 1$. Since $\deg(\pi_i(\Gamma)) = 3$ for all i , we have $\deg(\pi_i(\beta)) = 2$ for $i = 1, \dots, k - 1$. This implies that the minimal sub-scheme α of

Γ' such that $p' \in \langle \nu'(\alpha) \rangle$ is such that $\alpha \neq \beta$ where β is the degree 2 sub-scheme of Γ' . Now let $S' \subset (\mathbb{P}^1)^{k-1}$ be the projection by τ_k of the set of points of $S \subset \mathcal{S}(p)$ that are not in D , i.e. $S' := \tau_k(S \setminus S \cap D)$. Since $\sharp(S') \leq k-2$ and $p' \in \langle \nu'(\Gamma') \rangle$, the inductive assumption gives $\alpha \neq \Gamma'$ (it works even when $k=4$). Hence $\alpha = \{o'\}$. Thus $p \in \langle \nu(\{o\} \cup D) \rangle$. Hence $\dim(\langle \nu(\Gamma \cup D) \rangle) \leq \dim(\langle \nu(D) \rangle) + 2$, contradicting Claim 1. \square

Proof of Theorem 1.8: If $x=3$, then we may take as p a general point of $\sigma_3(X)$. Now assume $x \geq 4$ and hence $k \geq 5$. Apply Proposition 1.8 to $(\mathbb{P}^1)^{x+1}$ and then use Autarky (Lemma 2.4). \square

Question 3.2. Assume $k \geq 3$ and $n_i \geq 2$ for all i . Let $\Gamma \subset (\mathbb{P}^1)^k$ be a connected and curvilinear zero-dimensional scheme such that $\deg(\Gamma) = 3$ and for all i $\pi_i(\Gamma)$ is not contained in a line. Let $X \subset \mathbb{P}^r$ with $r = -1 + \prod_{i=1}^k (n_i + 1)$, be the Segre embedding of $(\mathbb{P}^1)^k$. Is $r_X(p) = 2k-1$ for a general $p \in \langle \nu(\Gamma) \rangle$?

By Autarky, Question 3.2 is true if and only if it is true when $n_i = 2$ for all i . Note that any two different Γ 's are in the same orbit for the group $\text{Aut}(\mathbb{P}^{n_1}) \times \cdots \times \text{Aut}(\mathbb{P}^{n_k})$ (in the set up of the proof of Theorem 1.7 these schemes Γ 's correspond to the case in which each $f_i : \mathbb{P}^1 \rightarrow \mathbb{P}^{n_i}$ is an embedding with $f_i(\mathbb{P}^1)$ a smooth conic). Hence the case $k=3$ of Question 3.2 is true by [9, Theorem 1.8].

4. PROOF OF THEOREM 1.10

Lemma 4.1. Fix an integer $c > 0$ and $u \in \mathbb{P}^1$. Let $E = cu \subset \mathbb{P}^1$ be the degree c effective divisor of \mathbb{P}^1 with u as its support. Let $g : E \rightarrow \mathbb{P}^n$ be any morphism. Then there is a non-negative integer $e \leq c$ and a morphism $h : \mathbb{P}^1 \rightarrow \mathbb{P}^n$ such that $h^*(\mathcal{O}_{\mathbb{P}^n}(1)) \cong \mathcal{O}_{\mathbb{P}^1}(e)$ and $h|_E = g$.

Proof. Every line bundle on E is trivial. We fix an isomorphism between $g^*(\mathcal{O}_{\mathbb{P}^n}(1))$ and $\mathcal{O}_E(c)$. After this identification, g is induced by $n+1$ sections u_0, \dots, u_n of $\mathcal{O}_E(c)$ such that at least one of them has a non-zero restriction at $\{u\}$. The map $H^0(\mathcal{O}_{\mathbb{P}^1}(c)) \rightarrow H^0(\mathcal{O}_E(c))$ is surjective and its kernel is the section associated to the divisor cu . Hence there are $v_0, \dots, v_n \in H^0(\mathcal{O}_{\mathbb{P}^1}(c))$ with $v_i|_E = u_i$ for all i . Not all sections v_0, \dots, v_n vanish at 0. If they have no common zero, then they define a morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^n$ extending g and we may take $e = c$. Now assume that they have a base locus and call F the scheme-theoretic base locus of the linear system associated to v_0, \dots, v_n . We have $\deg(F) \leq c$. Set $e := c - \deg(F)$ and $S := F_{\text{red}}$. The sections v_0, \dots, v_n induce a morphism $f : \mathbb{P}^1 \setminus S \rightarrow \mathbb{P}^n$ with $f|_E = g$. See v_0, \dots, v_n as elements of $|\mathcal{O}_{\mathbb{P}^1}(c)|$ and set $u_i := u - F \in |\mathcal{O}_{\mathbb{P}^1}(e)|$. By construction the linear system spanned by u_0, \dots, u_n has no base points, hence it induces a morphism $h : \mathbb{P}^1 \rightarrow \mathbb{P}^n$ such that $h^*(\mathcal{O}_{\mathbb{P}^n}(1)) \cong \mathcal{O}_{\mathbb{P}^1}(e)$. We have $h|_{\mathbb{P}^1 \setminus S} = f$ and hence $h|_E = g$. \square

Proof of Theorem 1.10: Let $E \subset \mathbb{P}^1$ be as in Lemma 4.1 and let E_1, \dots, E_α be the connected components of E . By assumption there is $p_i \in \langle \nu_{d_1, \dots, d_k}(E_i) \rangle$ such that $p \in \langle \{p_1, \dots, p_\alpha\} \rangle$. Note that if Theorem 1.10 is true for each (E_i, p_i) , then it is true for E . Hence it is sufficient to prove Theorem 1.10 under the additional assumption that E is connected, so from now on we assume

E connected.

Moreover, since $r_X(p) = 1 \leq 2 - 1 + \sum_i d_i$ if $c=1$, we may also assume that

$$\deg E = c \geq 2.$$

Finally, since the real-valued function $x \mapsto x \left(-1 + \sum_{i=1}^k d_i \right)$ is increasing for $x \geq 1$, with no loss of generality we may assume that, for any $G \subsetneq Z$,

$$p \notin \langle \nu_{d_1, \dots, d_k}(G) \rangle.$$

Fix $u \in \mathbb{P}^1$ and let $E = cu \subset \mathbb{P}^1$ be the degree c effective divisor of \mathbb{P}^1 with u as its support. Since Z is curvilinear and $\deg(Z) = c$, we have $Z \cong E$ as abstract zero-dimensional schemes. Let $g : E \rightarrow \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be the composition of an isomorphism $E \rightarrow Z$ with the inclusion $Z \hookrightarrow \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$:

$$g : E \rightarrow Z \hookrightarrow \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}.$$

Set $g_i := \pi_i \circ g$. If we apply Lemma 4.1 to each g_i , we get the existence of an integer $c_i \in \{0, \dots, c\}$ and of a morphism $h_i : \mathbb{P}^1 \rightarrow \mathbb{P}^{n_i}$ such that $h_{i|Z} = g_i$ and $h_i^*(\mathcal{O}_{\mathbb{P}^{n_i}}(1)) \cong \mathcal{O}_{\mathbb{P}^1}(c_i)$. The map

$$h = (h_1, \dots, h_k) : \mathbb{P}^1 \rightarrow \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k} \quad (6)$$

has multi-degree (c_1, \dots, c_k) . The curve

$$D := h(\mathbb{P}^1)$$

is an integral rational curve containing Z . Since $p \in \langle \nu_{d_1, \dots, d_k}(Z) \rangle$, we have $r_X(p) \leq r_{\nu_{d_1, \dots, d_k}(D)}(p)$. Thus it is sufficient to prove that $r_{\nu_{d_1, \dots, d_k}(D)}(p) \leq 2 + c \left(-1 + \sum_{i=1}^k d_i \right)$. Since $c_i \leq c$ for all i , it is sufficient to prove that $r_{\nu_{d_1, \dots, d_k}(D)}(p) \leq 2 - c + \sum_{i=1}^k c_i d_i$.

Set $\tilde{D} := \nu_{d_1, \dots, d_k}(D)$, $\tilde{Z} := \nu_{d_1, \dots, d_k}(Z)$, $m := \dim(\langle \tilde{D} \rangle)$ and

$$f = \nu_{d_1, \dots, d_k} \circ h : \mathbb{P}^1 \rightarrow \mathbb{P}^N.$$

By assumption \tilde{Z} is linearly independent in $\langle \tilde{D} \rangle \cong \mathbb{P}^m$ and in particular $c \leq m + 1$.

- (a) Assume that the map h defined in (6) is birational onto its image. The curve $\tilde{D} \subset \mathbb{P}^N$ just defined is a rational curve of degree $a := \sum_{i=1}^k c_i d_i$ contained in the projective space $\mathbb{P}^m := \langle \tilde{D} \rangle$ and non-degenerate in \mathbb{P}^m . Note that $a \geq m$ and that $p \in \langle \tilde{Z} \rangle$.
- (1) First assume that $a = m$. In this case \tilde{D} is a rational normal curve of \mathbb{P}^m . If $c \leq \lceil (a+1)/2 \rceil$, then Sylvester's theorem implies that $r_{\tilde{D}}(p) = a + 2 - c = 2 - c + \sum_{i=1}^k c_i d_i$. Now assume $c > \lceil (a+1)/2 \rceil$. Since \tilde{Z} is connected and curvilinear and $p \notin \langle \tilde{G} \rangle$ for any $G \subsetneq \tilde{Z}$, Sylvester's theorem implies $r_{\tilde{D}}(p) \leq c$.
- (2) Now assume $m < a$. There is a rational normal curve $C \subset \mathbb{P}^a$ and a linear subspace $W \subset \mathbb{P}^a$ such that $\dim(W) = a - m - 1$, $C \cap W = \emptyset$ and h is the composition of the degree a complete embedding $j : \mathbb{P}^1 \hookrightarrow \mathbb{P}^a$ and the linear projection $\ell : \mathbb{P}^a \setminus W \rightarrow \mathbb{P}^m$ from W . The scheme $E' := j(E)$ is a degree c curvilinear scheme and ℓ maps E' isomorphically onto \tilde{Z} . Since \tilde{Z} is linearly independent, then $\langle E' \rangle \cap W = \emptyset$ and ℓ maps isomorphically $\langle E' \rangle$ onto $\langle \tilde{Z} \rangle$. Thus there is a unique $q \in \langle E' \rangle$ such that $\ell(q) = p$. Take any finite set $S \subset j(\mathbb{P}^1)$ with $q \in \langle S \rangle$. Since $C \cap W = \emptyset$, $\ell(S)$ is a well-defined subset of \tilde{D} with cardinality $\leq \#(S)$. Hence $r_{\tilde{D}}(p) \leq r_C(q)$. As in step (a1) we see that either $r_C(q) = a + 2 - c$ (case $c \leq \lceil (a+1)/2 \rceil$) or $r_C(q) \leq c$ (case $c > \lceil (a+1)/2 \rceil$).
- (b) Now assume that h is not birational onto its image, but it has degree $k \geq 2$. Note that k divides c_i for all i . In this case we will prove that $r_{\nu_{d_1, \dots, d_k}(D)}(p) \leq 2 - c + \sum_{i=1}^k c_i d_i / k$. Let $h' : \mathbb{P}^1 \rightarrow h(\mathbb{P}^1)$ denote the normalization map. There is a degree k map $h'' : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ such that h is the composition of $h' \circ h''$ and the inclusion $h(\mathbb{P}^1) \subset \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$. We have $Z = h'(E')$, where $E' = cu'$ and $u' = h''(u)$. We use E' and h' instead of E and h and repeat verbatim step (a).

□

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